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Abstract

In the present paper, the existence of maximal elements for binary relations are studied. Generalized lexicographic relations are introduced and some results on existence of maximal elements are provided. A simple example shows that economies with "lexicographic consumers" need not have equilibria even though demand functions associated with generalized lexicographic relations may be continuous for positive prices.

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1 Introduction

Agents often act according to lexicographic preferences as demonstrated by numerous papers, see e.g. Colman and Stirk (1999) (using experiments with multi-attribute choice) and Lockwood (1999) (describing the structure of value expressions for Australian native forests) for recent empirical documentation.

That agents in general perform sequential decision procedures has long been recognized. In Georgescu-Roegen (1954) such agents are referred to as *homo economics B* using the following illustration: Assume that the agents have wants for food, palatable taste, and for entertaining friends. Moreover, assume that choice is between butter x_1 and margarine x_2. The want for food is reflected by the amount of calories $k = x_1 + x_2$ and palatable taste is offered only by butter. The agent may now act according to the following reasoning. The want for food is more important than the want for taste up to some limit $\bar{k}$, that is, the consumption choice is made according to the size of k for $k \leq \bar{k}$ and only if two combinations have the same amount of calories k the one with the largest amount of butter x_2 will be preferred. Now, if $k > \bar{k}$ the want for food is satisfied and choice will be made on the basis of the greatest value of x_2. For a common value of x_2, the want for entertaining friends may be shaped according to the number of friends who simply need food or those who can enjoy taste, that is according to the size of $e = ax_1 + bx_2$.

Sequential reasoning is also known from moral philosophy. In Rawls (1971) decision makers in the original position are supposed to choose principles for a just society. Rawls argues that such principles will be lexicographically (or *serially* to use his own term) ordered in the sense that freedom outranks welfare. Thus, *a departure from the institutions of equal liberty ... cannot be justified by, or compensated for, by greater social and economic advantages.* (p. 61).

For literature which focuses on lexicographic orders see e.g. Fishburn

(1975) for an axiomatic characterization, Chipman (1960),(1971) for a foundation of utility, or more recently Knoblauch (2000) who provides a set of necessary and sufficient conditions for the existence of an order homomorphism from a given preference relation to the lexicographic order on Euclidian n-space. Thereby she implicitly gives sufficient conditions for existence of maximal elements.

In the present paper we consider preference relations that are consistent with sequential decision procedures as mentioned above and we prove the existence of maximal elements.

From the literature on existence of maximal elements for binary relations, it is well known that the existence of continuous utility functions is not necessary in order to show that the demand set is non-empty. Transitivity has been weakened by considering acyclic relations along the lines of Bergstrom (1975) and Walker (1977). Moreover, assuming convexity of the set of alternatives and the upper contour sets has been the approach of Sonnenschein (1971), Yannelis & Prabhaker (1983) and Tian (1993). Lately Llinares (1998) considers a unified approach.

The generalized lexicographic relations considered in this paper do not satisfy the usual continuity conditions and hence, the standard theorems mentioned above do not apply to such relations. However, as we shall demonstrate, we may indeed obtain similar results concerning maximal elements for generalized lexocigraphic relations by suitable adjustments of the continuity conditions.

Finally, we consider a simple example of an economy with continuous demand and lexicographic preferences for which there are no equilibria.

2 Results

Suppose that X is a separable and connected topological space and that $P \subset X \times X$ is a relation. Let $U(x) = \{y \in X | (x, y) \in P\}$ be the set of alternatives that are strictly preferred to x, i.e. the upper contour set and let $U^{-1}(y) = \{x \in X | (x, y) \in P\}$ be the lower contour set for all $x \in X$. Then x is P-maximal provided that $U(x) = \emptyset$. Finally, denote by *co* A the convex hull of $A \subset X$.

Definition 1 *The relation $P \subset X \times X$ is a generalized lexicographic relation provided that $U(x) = \cup_{n \in \mathbf{N}} G_n(x)$ such that:*

- *for all $x \in X$ and $n \in \mathbf{N}$, $G_{n+1}(x) \subset \{y \in X | G_n(y) \subset G_n(x)\}$, and;*
- *for all $x \in X$, $G_1^{-1}(x)$ is open and $G_{n+1}^{-1}(x)$ is open in $\{y \in X | G_n(y) = G_n(x)\}$.*

Example: Let $P \subset \mathbf{R}^k \times \mathbf{R}^k$ be defined by $U(x) = \cup_{n \in \mathbf{N}} G_n(x)$ where

$$G_n(x) \;=\; \{y \in \mathbf{R}^k | y_1 = x_1, \ldots, y_{n-1} = x_{n-1} \text{ and } y_n > x_n\}$$

for all $n \leq k$ and $G_n(x) = \emptyset$ for all $n \geq k+1$. Then $P \subset \mathbf{R}^k \times \mathbf{R}^k$ is the lexicographic ordering of $\mathbf{R}^k$. Hence,

$$\begin{aligned} G_{n+1}(x) &\subset \{y \in \mathbf{R}^k | G_n(y) \subset G_n(x)\} \\ &= \{y \in \mathbf{R}^k | y_1 = x_1, \ldots, y_{n-1} = x_{n-1} \text{ and } y_n \geq x_n\} \end{aligned}$$

for $n \leq k$. Also, $G_1^{-1}(x)$ is open and $G_{n+1}^{-1}(x)$ is open in $\{y \in X | G_n(y) = G_n(x)\}$.

End of example

The first result on existence of maximal elements of generalized lexicographic relations is a variation of results by Bergstrom (1975) and Walker (1977).

Theorem 1 *Suppose that X is nonempty and compact and that P is a generalized lexicographic relation which satisfies:*

- *for all* $x_1, \ldots, x_k \in X$*, if* $x_2 \in G_n(x_1), \ldots, x_k \in G_n(x_{k-1})$ *then* $x_1 \notin G_n(x_k)$*.*

Then the set of P*-maximal elements is nonempty and compact.*

Proof: Let

$$M_n \;=\; \{x \in X | \cup_{j=1}^{n} G_j(x) = \emptyset\}$$

for all $n \in \mathbf{N}$. Then $M_{n+1} \subset M_n$ for all $n \in \mathbf{N}$. Now, M_1 is nonempty and compact according to Bergstrom (1975) and Walker (1977). Moreover, M_{n+1} is nonempty and compact provided that M_n is nonempty and compact because M_n has the same properties as X and G_{n+1} has the same properties on M_n as G_1 has on X. Therefore

$$\{x \in X | U(x) = \emptyset\} \;=\; \cap_{n \in \mathbf{N}} M_n$$

is nonempty and compact because M_n is nonempty and compact, $M_{n+1} \subset M_n$ for all $n \in \mathbf{N}$ and X has the finite intersection property.

Q.E.D

The second result on existence of maximal elements of generalized lexicographic relations is a variation of a result by Sonnenschein (1971).

Theorem 2 *Suppose that* X *is compact and convex and that* P *is a generalized lexicographic relation which satisfies:*

- $x \notin co\; G_n(x)$*, and;*

- $\{y \in X | G_n(y) \subset G_n(x)\}$ *is convex.*

Then the set of P*-maximal elements is nonempty, compact and convex.*

Proof: Let

$$M_n \;=\; \{x \in X | \cup_{j=1}^{n} G_j(x) = \emptyset\}.$$

Then $M_{n+1} \subset M_n$ for all $n \in \mathbf{N}$. Now, M_1 is nonempty and compact according to Sonnenschein (1971) and M_1 is convex because $\{y \in X | G_n(y) \subset G_n(x)\}$ is convex. Moreover, M_{n+1} is nonempty, convex and compact provided that M_n is nonempty, convex and compact because M_n has the same properties as X and G_{n+1} has the same properties on M_n as G_1 has on X. Therefore

$$\{x \in X | U(x) = \emptyset\} \;=\; \cap_{n \in \mathbf{N}} M_n$$

is nonempty and compact because M_n is nonempty and compact, $M_{n+1} \subset M_n$ for all $n \in \mathbf{N}$ and X has the finite intersection property.

Q.E.D

Finally, total and transitive relations that are representable by continuous utility functions are considered.

Theorem 3 *Suppose that $P \subset X \times X$ is a total and transitive relation and that $(P_n)_{n \in \mathbf{N}} \subset X \times X$ is a sequence of total, transitive and continuous relations such that $U(x) = \cup_{n \in \mathbf{N}} F_n(x)$ where $F_1(x) = U_1(x)$ and $F_{n+1}(x) = U_{n+1}(x) \cap \{y \in X | y \in F_n(x)\}$. Then:*

- *there exists a sequence of continuous functions, $(u_n)_{n \in \mathbf{N}} : X \to \mathbf{R}$, such that $(x, y) \in P$ if and only if*

  $$\min\{n | u_n(x) > u_n(y)\} \;\leq \min\{n' | u_{n'}(y) > u_{n'}(x)\},$$

  and;

- *if X is compact then the set of P-maximal elements is nonempty and compact.*

Proof: The first part of the theorem follows directly from Debreu (1954) and the second part of the theorem follows directly from Theorem 1.

Q.E.D

3 An Example

Given the results above on the existence of maximal elements it is tempting to suggest that economies for which consumers have generalized lexicographic preferences, and demand is continuous for positive prices, will always have equilibria. This, however, is not necessarily the case as the following simple example shows.

Consider an economy with one consumer. The comsuption possibility set is given by $X = \mathbf{R}^2_+$, initial endowments by $\omega = (1, 1)$ and preferences by the lexicographic relation $P(x) = \{y \in X | y_1 > x_1\} \cup \{y \in X | y_1 = x_1 \text{ and } y_2 > x_2\}$. Then excess demand is $e(p) = (p_2/p_1, -1)$ for all $p \in \mathbf{R}^2_{++}$ and therefore there is no equilibrium.

The lack of equilibrium may be seen as arising from the fact that it is not possible to separate ω and $P(\omega)$, i.e. there exists no $p \in \mathbf{R}^2$ such that $p{\cdot}\omega < p{\cdot}x$ for all $x \in P(\omega)$. However, for economies with several consumers, this view is not appropriate, there the lack of equilibrium may be seen as arising from the fact that excess demand does not have the "right" boundary behavior, i.e. $\lim_{p \to q} \|e(p)\| = \infty$ for all $q \in \partial\mathbf{R}^2_{++}$ where $q \neq 0$. Indeed, $\lim_{p \to (1,0)} e(p) = (0, -1)$ so $\lim_{p \to (1,0)} \|e(p)\| = 1$. For economies in which excess demand for least one consumer have the "right" boundary behavior equilibria exists because aggregate excess demand inherits the "right" boundary behavior from this consumer.

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